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LETTER TO THE EDITOR

Integrable XYZ spin chain with boundaries

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Abstract. We consider a general class of boundary terms of the open XYZ spin-$\frac{1}{2}$ chain compatible with integrability. We have obtained the general elliptic solution of a K-matrix obeying the boundary Yang–Baxter equation using the R-matrix of the eight-vertex model and derived the associated integrable spin-chain Hamiltonian.

$1+1$-dimensional integrable models with boundaries find interesting applications in particle physics as well as condensed-matter systems. In view of this, attempts have recently been made at the integrable extension of conformal field theories [1, 2], both massive and massless integrable quantum field theories [3–6] and solvable lattice models [7–13] to those with boundary terms. In the case of lattice models, relying on the earlier work by Cherednik [8], Sklyanin has given [9] a general framework which enables us to treat this problem on an algebraic footing. In particular, the general solution of integrable boundary terms has been found in the XXZ and the XXX Heisenberg spin-chain system [12] based on his framework. In this letter, we will evaluate the general solution of boundary interactions in the case of XYZ Heisenberg spin-chain systems.

The Hamiltonian of the XYZ spin-$\frac{1}{2}$ chain is given by the transfer matrix of the eight-vertex model [14]. The eight-vertex model is defined in terms of the Boltzmann weights given by the elliptic solution $R(u)$ of the Yang–Baxter (YB) equation

$$R_{12}(u-u')R_{13}(u)R_{23}(u') = R_{23}(u')R_{13}(u)R_{12}(u-u'). \tag{1}$$

Here, we regard $R(u)$ as linear operators acting on the tensor product of vector spaces $V\otimes V$ with $V = \boldsymbol{C}v_+ \oplus \boldsymbol{C}v_-$ and $R_{12} \equiv R\otimes 1$, $R_{23} \equiv 1\otimes R$ etc as those acting on $V_1\otimes V_2\otimes V_3$, where $V_i \cong V$, $i = 1, 2, 3$. Setting

$$R(u)v_{\varepsilon_1'}\otimes v_{\varepsilon_2'} = \sum_{\varepsilon_1,\varepsilon_2} v_{\varepsilon_1}\otimes v_{\varepsilon_2} R(u)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}$$

and arranging the elements of R in the order $(\varepsilon_1, \varepsilon_2) = (++), (+-), (-+), (--)$, one can express the eight-vertex R-matrix as follows

$$R(u) = \begin{pmatrix} \operatorname{sn}(u+\eta) & 0 & 0 & k\operatorname{sn}\eta\operatorname{sn}u\operatorname{sn}(u+\eta) \\ 0 & \operatorname{sn}u & \operatorname{sn}\eta & 0 \\ 0 & \operatorname{sn}\eta & \operatorname{sn}u & 0 \\ k\operatorname{sn}\eta\operatorname{sn}u\operatorname{sn}(u+\eta) & 0 & 0 & \operatorname{sn}(u+\eta) \end{pmatrix} \tag{2}$$

where $\operatorname{sn}u \equiv \operatorname{sn}(u;k)$ is the Jacobi elliptic function of modulus $0 \leqslant k \leqslant 1$.

Let $\mathcal{P}_{ij}$ be the transposition operator on $V_i \otimes V_j$, i.e. $\mathcal{P}(x \otimes y) = y \otimes x$. R-matrix (2) is known to have the following desirable properties:

Regularity $\quad R(0) = r(\eta)\mathcal{P} \qquad r(\eta) = \operatorname{sn}\eta \qquad (3)$

P-symmetry $\quad \mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12} = R_{12}(u) \qquad (4)$

T-symmetry $\quad R_{12}^{t_1 t_2}(u) = R_{12}(u) \qquad (5)$

Unitarity $\quad R_{12}(u)R_{12}(-u) = \rho(u)1 \qquad \rho(u) = \operatorname{sn}^2\eta - \operatorname{sn}^2 u \qquad (6)$

Crossing unitarity $\quad R_{12}^{t_1}(u)R_{12}^{t_1}(-u-\eta) = \tilde{\rho}(u)1 \qquad \tilde{\rho}(u) = \operatorname{sn}^2\eta - \operatorname{sn}^2(u+\eta). \quad (7)$

In the case of periodic boundary conditions, it is known that the YB equation (1) implies a commuting family of transfer matrices. Hence, the model is integrable.

We now consider the eight-vertex model with boundary interactions. Aiming at describing integrable systems with boundaries, Sklyanin [9] has introduced a pair of matrices $K_+(u)$ and $K_-(u)$. The effects of the presence of boundaries at the left and right ends are solely described by $K_+(u)$ and $K_-(u)$, respectively. $K_\pm(u)$ are defined as the solutions to the relations

$$R_{12}(u-u')\overset{1}{K}_-(u)R_{12}(u+u')\overset{2}{K}_-(u') = \overset{2}{K}_-(u')R_{12}(u+u')\overset{1}{K}_-(u)R_{12}(u-u') \tag{8}$$

$$R_{12}(-u+u')\overset{1}{K}{}^{t_1}_+(u)R_{12}(-u-u'-2\eta)\overset{2}{K}{}^{t_2}_+(u') = \overset{2}{K}{}^{t_2}_+(u')R_{12}(-u-u'-2\eta)\overset{1}{K}{}^{t_1}_+(u)R_{12}(-u+u') \tag{9}$$

where $\overset{1}{K}_\pm \equiv K_\pm \otimes \mathrm{id}_{V_2}$ and $\overset{2}{K}_\pm \equiv \mathrm{id}_{V_1} \otimes K_\pm$. Equations (8) and (9) are called boundary YB equations and $K_\pm(u)$ are called boundary K-matrices.

The boundary YB equations imply a commuting family of transfer matrices [9]. The transfer matrix $t(u)$, in this case, is defined using the $K_\pm$ and the monodromy matrix $T(u)$ as

$$t(u) = \operatorname{tr}[K_+(u)T(u)K_-(u)T^{-1}(-u)] \tag{10}$$

where $T(u)$ is given by

$$T(u) = R_{N0}(u)R_{N-10}(u)\cdots R_{10}(u). \tag{11}$$

The trace in (10) should be taken over V_0. Then, the commuting property of $t(u)$

$$[t(u), t(u')] = 0 \tag{12}$$

follows from the properties of R and the boundary YB equations (8) and (9).

The problem is now to solve equations (8) and (9) and find general solutions for K_- and K_+ using the eight-vertex R-matrix given in (2). It suffices to consider the first equation, because of the following fact. Suppose $K_-(u)$ is a solution of the first equation, then the function

$$K_+(u) = K_-^t(-u-\eta) \tag{13}$$

gives the solution for the second equation.

We now proceed to solve equation (8). Write $K_-(u)$ as

$$K_-(u) = \begin{pmatrix} x(u) & y(u) \\ z(u) & v(u) \end{pmatrix}. \tag{14}$$

We have found that, out of the sixteen equations in boundary YB equation (8), only three are independent:

$$s_- v v' + s_+ x v' = s_+ v x' + s_- x x' \tag{15}$$

$$y z' + k s_- s_+ z z' = z y' + k s_- s_+ y y' \tag{16}$$

$$\begin{aligned} &S_- s_+ y x' + k s_\eta^2 S_- s_- z x' + k s_\eta S_- S_+ (s_- v z' + s_+ x z') \\ &\qquad = S_+ s_- y x' + k s_\eta^2 S_+ s_+ z x' + s_\eta (s_- v y' + s_+ x y') \end{aligned} \tag{17}$$

where we set $x \equiv x(u)$, $x' \equiv x(u')$ etc, and

$$s_\eta \equiv \operatorname{sn} \eta \qquad s_\pm \equiv \operatorname{sn}(u \pm u') \qquad S_\pm \equiv \operatorname{sn}(u \pm u' + \eta). \tag{18}$$

In the following, we also use the notation $\alpha(u) \equiv v(u)/x(u)$, $\beta(u) \equiv z(u)/y(u)$ and $\gamma(u) \equiv y(u)/x(u)$.

Dividing (15) by xx', one obtains

$$\alpha(u') = \frac{\operatorname{sn}(u+u')\alpha(u) + \operatorname{sn}(u-u')}{\operatorname{sn}(u-u')\alpha(u) + \operatorname{sn}(u+u')}. \tag{19}$$

Taking the limit $u' \to u$ of $\alpha(u') - \alpha(u)/u' - u$, one obtains the following differential equation:

$$\frac{\mathrm{d}\alpha(u)}{\mathrm{d}u} = -\frac{1-\alpha(u)^2}{\operatorname{sn} 2u}. \tag{20}$$

After the change of variable $t = \operatorname{sn} u$, the integration of (20) takes the form

$$\int \frac{\mathrm{d}\alpha}{1-\alpha^2} = -\frac{1}{2}\int \mathrm{d}t \frac{1-k^2t^4}{t(1-t^2)(1-k^2t^2)} \tag{21}$$

where we have used the formulae

$$\operatorname{sn}(2u) = \frac{2\operatorname{sn}(u)\operatorname{cn}(u)\operatorname{dn}(u)}{1-k^2\operatorname{sn}^4(u)} \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{sn}(u) = \operatorname{cn}(u)\operatorname{dn}(u). \tag{23}$$

One can easily obtain the general solution of equation (21)

$$\frac{v(u)}{x(u)} = \frac{C\operatorname{cn} u \operatorname{dn} u - \operatorname{sn} u}{C\operatorname{cn} u \operatorname{dn} u + \operatorname{sn} u} \tag{24}$$

where C is an arbitrary constant.

In a similar way, from (16) one obtains

$$\beta(u') = \frac{\beta(u) + k\,\mathrm{sn}(u+u')\,\mathrm{sn}(u-u')}{k\,\mathrm{sn}(u+u')\,\mathrm{sn}(u-u')\beta(u) + 1} \tag{25}$$

and the differential equation

$$\frac{\mathrm{d}\beta(u)}{\mathrm{d}u} = -k\,\mathrm{sn}\,2u(1-\beta(u)^2). \tag{26}$$

This implies the general solution

$$\frac{z(u)}{y(u)} = \frac{\lambda(1-k\,\mathrm{sn}^2 u) - 1 - k\,\mathrm{sn}^2 u}{\lambda(1-k\,\mathrm{sn}^2 u) + 1 + k\,\mathrm{sn}^2 u} \tag{27}$$

where λ is another arbitrary constant.

Dividing (17) by yy', the third equation (17) can be written

$$\frac{\gamma(u)}{\gamma(u')} = \frac{s_\eta(s_-\alpha(u) + s_+)(1 - ks_+s_-\beta(u'))}{S_-s_+ - S_+s_- + k\beta(u)s_\eta^2(S_-s_- - S_+s_+)}. \tag{28}$$

Substituting (25) into (28) and

$$\frac{\gamma(u)}{\gamma(u')} = \frac{1}{\mathrm{sn}\,2u'}\frac{\mathrm{sn}(u-u')\alpha(u) + \mathrm{sn}(u+u')}{1 + k\beta(u)\,\mathrm{sn}(u-u')\,\mathrm{sn}(u+u')} \tag{29}$$

and replacing $\alpha(u)$ and $\beta(u)$ by the right-hand side of (24) and (27), one can factorize (28) in the form of the ratio of the same functions; one with argument u and the other with u', respectively. We thus find

$$\frac{y(u)}{x(u)} = \mu\frac{\lambda(1-k\,\mathrm{sn}^2 u) + 1 + k\,\mathrm{sn}^2 u}{C\,\mathrm{cn}\,u\,\mathrm{dn}\,u + \mathrm{sn}\,u} \tag{30}$$

where μ is the third arbitrary constant.

From (27) and (30), one can now obtain the following ratio:

$$\frac{z(u)}{x(u)} = \mu\frac{\lambda(1-k\,\mathrm{sn}^2 u) - 1 - k\,\mathrm{sn}^2 u}{C\,\mathrm{cn}\,u\,\mathrm{dn}\,u + \mathrm{sn}\,u}. \tag{31}$$

It is then not difficult to check that the above solutions satisfy all the remaining equations if one notes the identity

$$s_\eta^2\frac{S_+s_+ - S_-s_-}{S_-s_+ - S_+s_-} = \frac{S_+S_- - s_+s_-}{1 - k^2S_+S_-s_+s_-} = s_\eta\,\mathrm{sn}(2u+\eta). \tag{32}$$

In summary, we have obtained the general solution of (8) as $K_-(u) = K(u;\xi,\lambda,\mu)$ with

$$K(u;\xi,\lambda,\mu) = \frac{1}{\mathrm{sn}\,\xi}\begin{pmatrix} \mathrm{sn}(\xi+u) & \mu\,\mathrm{sn}\,2u\dfrac{\lambda(1-k\,\mathrm{sn}^2 u)+1+k\,\mathrm{sn}^2 u}{1-k^2\,\mathrm{sn}^2\xi\,\mathrm{sn}^2 u} \\ \mu\,\mathrm{sn}\,2u\dfrac{\lambda(1-k\,\mathrm{sn}^2 u)-1-k\,\mathrm{sn}^2 u}{1-k^2\,\mathrm{sn}^2\xi\,\mathrm{sn}^2 u} & \mathrm{sn}(\xi-u) \end{pmatrix} \tag{33}$$

where we have set $C = \mathrm{sn}\,\xi / \mathrm{cn}\,\xi\,\mathrm{dn}\,\xi$ and replaced $\mu\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi$ by μ†. We normalize the matrix $K_-(u)$ as $K_-(0) = 1$ for later convenience [9].

In the trigonometric limit $k \to 0$, where $\mathrm{sn}\,u \to \sin u$, we recover the result in the case of the six-vertex model given by de Vega and González Ruiz [12]. The rational limit is obtained from the trigonometric K-matrix by scaling $u \to \eta u$, $\xi \to \eta\xi$ and taking the limit $\eta \to 0$.

Let us next consider the corresponding XYZ spin-chain Hamiltonian. Because of equation (12), one can regard the transfer matrix $t(u)$ as the generating function of integrals of motion of the system. Its first logarithmic derivative implies the following Hamiltonian:

$$\begin{aligned} H &= 2r(\eta)t^{-1}(0)(t'(0) - \mathrm{tr}\,K'_+(0)) \\ &= 2r(\eta)\left(\sum_{n=1}^{N-1} H_{n,n+1} + \tfrac{1}{2}\overset{1}{K}{}'_-(0) + \frac{\mathrm{tr}_0\,\overset{0}{K}_+(0) H_{N0}}{\mathrm{tr}\,K_+(0)}\right) \end{aligned} \tag{34}$$

where the two-site Hamiltonian is given by

$$H_{n,n+1} = \frac{1}{r(\eta)}\mathcal{P}_{nn+1}R'_{nn+1}(0). \tag{35}$$

By a direct calculation with the K-matrices $K_-(u) = K_-(u; \xi_-, \lambda_-, \mu_-)$ and $K_+(u) = K_-^t(-u-\eta; -\xi_+, -\lambda_+, -\mu_+)$ together with R-matrix (2), one obtains the following result:

$$\begin{aligned} H = \sum_{n=1}^{N-1}&((1+\Gamma)\sigma_n^x\sigma_{n+1}^x + (1-\Gamma)\sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z) \\ &+ \mathrm{sn}\,\eta(A_-\sigma_1^z + B_-\sigma_1^+ + C_-\sigma_1^- + A_+\sigma_N^z + B_+\sigma_N^+ + C_+\sigma_N^-) \end{aligned} \tag{36}$$

where

$$\begin{gathered} \Gamma = k\,\mathrm{sn}^2\eta \qquad \Delta = \mathrm{cn}\,\eta\,\mathrm{dn}\,\eta \\ A_\pm = \frac{\mathrm{cn}\,\xi_\pm\,\mathrm{dn}\,\xi_\pm}{\mathrm{sn}\,\xi_\pm} \qquad B_\pm = \frac{2\mu_\pm(\lambda_\pm + 1)}{\mathrm{sn}\,\xi_\pm} \qquad C_\pm = \frac{2\mu_\pm(\lambda_\pm - 1)}{\mathrm{sn}\,\xi_\pm}. \end{gathered} \tag{37}$$

In conclusion, we have obtained the general elliptic solution of the boundary YB equation for the K-matrices and derived the Hamiltonian of the associated XYZ spin-$\frac{1}{2}$ chain with boundary terms.

An immediate question is to find the ground-state energy and the excitation spectrum of the XYZ Hamiltonian we have derived. The diagonalization of this Hamiltonian can be achieved by means of the generalized algebraic Bethe ansatz [9] with some modification as in the periodic boundary condition case [14, 15]. This subject is now under investigation.

It is shown using the results of Bethe-ansatz-type analysis that, by tuning the X–Y anisotropy coupling (Γ in equation (36)), the XYZ Hamiltonian with periodic boundary condition gives rise to the quantum sine–Gordon theory in the continuum limit [16]. In the case of the open XYZ spin chain, it is of interest to ask whether one can tune the coupling of boundary terms together with the Γ so that one can derive its field-theory limit. The

† The solution obtained in [12] for the XYZ model are the special cases of (33) associated with the special solutions $\alpha(u)^2 = 1$, $\beta(u)^2 = 1$ of equations (20) and (26).

resulting theory is expected to be the quantum boundary sine–Gordon theory [3, 4]. In the limit $N \to \infty$, we have three boundary terms proportional to σ_1^x, σ_1^y and σ_1^z associated with three free parameters in K_-, whereas the boundary term proposed by Sklyanin [17] and Ghoshal and Zamolodchikov [3] has two parameters. It is necessary to explain this difference in the analysis of the continuum limit.

Furthermore, the higher logarithmic derivatives of the commuting transfer matrix give an infinite number of conserved quantities. In the closed XYZ spin-chain case, it was shown [18] that the conservation laws associated with these quantities yield selection rules in the scattering process of the quantum sine–Gordon theory. It is an interesting question to ask how the parameters appearing in the K-matrices affect the scattering process of the boundary sine–Gordon theory.

We will present our result on these problems in future publications.

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